Oxford Cambridge and RSA

Wednesday 22 May 2024 – Afternoon

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

You must have:
- the Printed Answer Booklet
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

QP

INSTRUCTIONS
- Use black ink. You can use an HB pencil, but only for graphs and diagrams.
- Write your answer to each question in the space provided in the **Printed Answer Booklet**. If you need extra space use the lined pages at the end of the Printed Answer Booklet. The question numbers must be clearly shown.
- Fill in the boxes on the front of the Printed Answer Booklet.
- Answer **all** the questions.
- Where appropriate, your answer should be supported with working. Marks might be given for using a correct method, even if your answer is wrong.
- Give your final answers to a degree of accuracy that is appropriate to the context.
- Do **not** send this Question Paper for marking. Keep it in the centre or recycle it.

INFORMATION
- The total mark for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- This document has **8** pages.

ADVICE
- Read each question carefully before you start your answer.

Section A (36 marks)

1 By expressing $\frac{1}{r+1}-\frac{1}{r+2}$ as a single fraction, find $\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)}$ in terms of n. **[4]**

2 Two complex numbers are given by $u=-1+\text{i}$ and $v=-2-\text{i}$.

(a) **(i)** Find $u-v$ in the form $a+b\text{i}$, where a and b are real. **[1]**

(ii) **In this question you must show detailed reasoning.**

Find $\frac{u}{v}$ in the form $a+b\text{i}$, where a and b are real. **[3]**

(b) Express u in exact modulus-argument form. **[3]**

3 The equation $2x^3-2x^2+8x-15=0$ has roots α, β and γ.

Determine the value of $\alpha^2+\beta^2+\gamma^2$. **[4]**

4 The equation of a curve is $y=\frac{1}{\sqrt{k^2+x^2}}$, where k is a positive constant. The region between the x-axis, the y-axis and the line $x=k$ is rotated through 2π radians about the x-axis.

Given that the volume of the solid of revolution formed is $1\,\text{unit}^3$, find the exact value of k. **[4]**

5 **(a)** Given that $\mathbf{u}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$, $\mathbf{v}=\begin{pmatrix}a\\0\\1\end{pmatrix}$ and $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}1\\b\\3\end{pmatrix}$, find a and b. **[3]**

(b) Using $\mathbf{u}\times\mathbf{v}$, determine the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, given that this angle is acute. **[3]**

6 On separate Argand diagrams, sketch the set of points represented by each of the following.

(a) $|z-1-2\mathrm{i}| \leqslant 4$. **[3]**

(b) $\arg(z+\mathrm{i}) = \frac{1}{3}\pi$. **[3]**

7 (a) Explain why $\int_1^2 \frac{1}{\sqrt[3]{x-2}}\,\mathrm{d}x$ is an improper integral. **[1]**

(b) **In this question you must show detailed reasoning.**

Use an appropriate limit argument to evaluate this integral. **[4]**

Section B (108 marks)

8 **(a)** Specify fully the transformation T of the plane associated with the matrix $\mathbf{M}$, where $\mathbf{M} = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and λ is a non-zero constant. **[2]**

(b) **(i)** Find $\det \mathbf{M}$. **[1]**

(ii) Deduce **two** properties of the transformation T from the value of $\det \mathbf{M}$. **[2]**

(c) Prove that $\mathbf{M}^n = \begin{pmatrix} 1 & n\lambda \\ 0 & 1 \end{pmatrix}$, where n is a positive integer. **[4]**

(d) Hence specify fully a **single** transformation which is equivalent to n applications of the transformation T. **[1]**

9 A curve has polar equation $r = a\sin 3\theta$, for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

(a) Sketch the curve. Indicate the parts of the curve where r is negative by using a broken line. **[3]**

(b) **In this question you must show detailed reasoning.**

Determine the area of one of the loops of the curve. **[5]**

10 **(a)** Write down the first three terms of the Maclaurin series for $\ln(1+x^3)$. **[1]**

(b) Use these three terms to show that $\ln(1.125) \approx \frac{n}{1536}$, where n is an integer to be determined. **[3]**

(c) Charlie uses the same first three terms of the series to approximate $\ln 9$ and gets an answer of 147, correct to 3 significant figures. However, $\ln 9 = 2.20$ correct to 3 significant figures.

Explain Charlie's error. **[2]**

11 The plane Π has equation $2x - y + 2z = 4$. The point P has coordinates (8, 4, 5).

(a) Calculate the shortest distance from P to Π. **[2]**

The line L has equation $\frac{x-2}{3} = \frac{y}{2} = \frac{z+3}{4}$.

(b) Verify that P lies on L. **[2]**

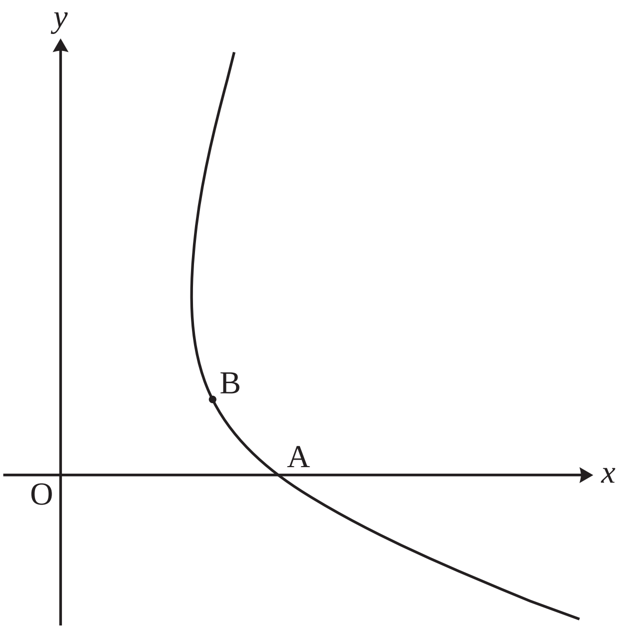
(c) Find the coordinates of the point of intersection of L and Π. **[3]**

(d) Determine the acute angle between L and Π. **[4]**

(e) Use the results of parts **(b)**, **(c)** and **(d)** to verify your answer to part **(a)**. **[3]**

12 The diagram shows the curve with parametric equations

$x = 2\cosh t + \sinh t,\ y = \cosh t - 2\sinh t$.



(a) The curve crosses the positive x-axis at A.

(i) Determine the value of the parameter t at A, giving your answer in logarithmic form. **[4]**

(ii) Find the x-coordinate of A, giving your answer correct to **3** significant figures. **[2]**

(b) The point B has parameter $t = 0$.

Determine the equation of the tangent to the curve at B. **[6]**

Turn over

13 The complex number z is defined as $z = \frac{1}{3}e^{i\theta}$ where $0 < \theta < \frac{1}{2}\pi$.

On an Argand diagram, the point O represents the complex number 0, and the points $P_1, P_2, P_3, \ldots$ represent the complex numbers $z, z^2, z^3, \ldots$ respectively.

(a) Write down each of the following.

(i) The ratio of the lengths $OP_{n+1} : OP_n$ **[1]**

(ii) The angle $P_{n+1}OP_n$ **[1]**

(b) **(i)** Show that $(3 - e^{i\theta})(3 - e^{-i\theta}) = a + b\cos\theta$, where a and b are integers to be determined. **[2]**

(ii) By considering the sum to infinity of the series $z + z^2 + z^3 + \ldots$, show that

$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \ldots = \frac{3\sin\theta}{10 - 6\cos\theta}.$$ **[6]**

14 **(a)** Find the general solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 12e^{-x}$. **[7]**

You are given that y tends to zero as x tends to infinity, and that $\frac{dy}{dx} = 0$ when $x = 0$.

(b) Find the exact value of x for which $y = 0$. **[5]**

15 Three planes have equations

$$\begin{aligned} x + ky + 3z &= 1, \\ 3x + 4y + 2z &= 3, \\ x + 3y - z &= -k, \end{aligned}$$

where k is a constant.

(a) Show that the planes meet at a point except for one value of k, which should be determined. **[4]**

(b) Show that, when the planes do meet at a point, the y-coordinate of this point is independent of k. **[6]**

16 **In this question you must show detailed reasoning.**

Show that $\int_0^1 \frac{1}{\sqrt{x^2+x+1}}\,dx = \ln\left(\frac{a+b\sqrt{3}}{c}\right)$, where a, b and c are integers to be determined. **[6]**

17 In an industrial process, a container initially contains 1000 litres of liquid. Liquid is drawn from the bottom of the container at a rate of 5 litres per minute. At the same time, salt is added to the top of the container at a constant rate of 10 grams per minute. After t minutes the mass of salt in the container is x grams, and you are given that $x = 0$ when $t = 0$.

In modelling the situation, it is assumed that the salt dissolves instantly and uniformly in the liquid, and that adding the salt does not change the volume of the liquid.

(a) **(i)** Show that the concentration of salt in the liquid after t minutes is $\frac{x}{1000-5t}$ grams per litre. **[1]**

(ii) Hence show that the mass of salt in the container is given by the differential equation

$$\frac{dx}{dt} + \frac{x}{200-t} = 10.$$ **[3]**

(b) Show by integration that $x = 10(200-t)\ln\left(\frac{200}{200-t}\right)$. **[8]**

(c) **(i)** Hence determine the mass of salt in the container when half the liquid is drawn off. **[2]**

(ii) Determine also the time at which the mass of salt in the container is greatest. **[5]**

(d) When the process is run, it is found that the concentration of salt over time is higher than predicted by the model.

Suggest a reason for this. **[1]**

END OF QUESTION PAPER

